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Solvable Lie algebras with Heisenberg ideals

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Abstract. All finite-dimensional indecomposable solvable Lie algebras $L(n,f)$, having the Heisenberg algebra $H(n)$ as the nilradical, are constructed. The number of non-nilpotent elements f that can be added to $H(n)$ satisfies $f \leqslant n+1$. The Casimir and generalized Casimir operators of the algebras $L(n,f)$ are obtained.

1. Introduction

The purpose of this article is to construct all indecomposable solvable Lie algebras L that have the $(2n+1)$-dimensional Heisenberg algebra $H(n)$ as their maximal nilpotent ideals. We also construct the Casimir invariants of L (polynomials in the enveloping algebra, commuting with all elements of L), and generalized Casimir operators (rational functions of the basis elements of L, commuting with all elements of L).

The Heisenberg algebras $H(n)$ with basis

$$\{P_1, \dots P_n, B_1, \dots, B_n, H\} \tag{1.1}$$

is of primordial importance in quantum mechanics. The operators P_i in this case correspond to linear momenta, B_i to coordinates and the central element H is proportional to the Planck constant. The extension of the algebra $H(n)$ by further operators $S_1, \dots, S_f$ is then a question of the algebra of quantum mechanical observables. We shall denote these extensions $L(n,f)$.

The Heisenberg algebra $H(n)$ is also a subalgebra of the quantum mechanical Galilei algebra [1,2] (or extended Galilei algebra). The operators P_i in this interpretation generate space translations, B_i generate Galilei boosts and H changes the phase of the wavefunction. The algebra $H(n)$ is also a subalgebra of the symmetry algebra of the heat equation, [3], and of the nonlinear Schrödinger equation with any nonlinearity $F(|\psi|)$, depending only on the absolute value of the wavefunction ψ and of many other partial differential equations occurring in non-relativistic physical theories [4,5]. In this context, extensions of the Heisenberg algebra $H(n)$ are part of a study of physical theories with symmetries going beyond translations and Galilei boosts.

An algebra that plays an important role in the microscopic theory of collective motions in nuclei is a semidirect product of the symplectic Lie algebra with the Heisenberg algebra as an ideal [6,7]

$$w\mathrm{sp}(2,\Re) = \mathrm{sp}(2n,\Re) \rhd H(n).$$

It turns out that all solvable indecomposable Lie algebras $L(n,f)$ obtained as extensions of the Heisenberg algebra $H(n)$ are subalgebras of $\mathrm{sp}(2n,\Re) \rhd H(n)$ (or $\mathrm{sp}(2n,C) \rhd H(n)$). They should hence have a role to play in the theory of nuclear collective motions.

From the mathematical point of view this investigation is part of a classification of all finite-dimensional Lie algebras. The Levi theorem [8–10] tells us that every finite-dimensional Lie algebra L is a semidirect sum of a semisimple Lie algebra and a solvable ideal (the radical $R(L)$). Semisimple Lie algebras over fields of characteristic zero have been classified by Cartan [11]. The classification of solvable Lie algebras is, however, complete only for low dimensions (dim $L \leqslant 6$). [12–14]. Malcev [15] has obtained important results on the structure of solvable Lie algebras, but has not classified all solvable Lie algebras with a given maximal nilpotent ideal (e.g. $H(n)$). A considerable literature exists on representations of solvable Lie algebras and groups; for a review see [16].

In section 2 we provide a classification of all indecomposable solvable Lie algebras $L(n,f)$ containing $H(n)$ as their maximal nilpotent ideal. In particular we show that f is restricted to $f \leqslant n+1$ and that we have $L(n,f) \subset \mathrm{sp}(2n,F) \rhd H(n)$.

All Casimir and generalized Casimir operators of $L(n,f)$ are obtained in section 3. The Casimir operators, when they exist, include the central element $H \in L(n,f)$ and second-order polynomials in the enveloping algebra of $L(n,f)$. In other cases, only generalized Casimir operators exist: they are second-order operators, multiplied by H^{-1}.

2. Classification of solvable Lie algebras with nilradical $H(n)$

2.1. *Preliminaries*

Let us first recall some well known results on solvable Lie algebras, that we shall need below [9, 10, 12]. We consider Lie algebras over a field F, with $F = \Re$, or $F = C$ (real or complex numbers).

A solvable Lie algebra L is characterized by the fact that its derived series: $L^0 \equiv L$, $L^1 = [L,L], \ldots, L^{j+1} = [L^j, L^j]$ terminates ($L^k = 0$ for some $k \in Z^{\geqslant 0}$).

A solvable Lie algebra is nilpotent if its lower central series $L^{(0)} = L$, $L^{(1)} = [L, L^{(0)}] \ldots, L^{(j+1)} = [L, L^{(j)}]$ terminates.

The nilradical $NR(L)$ of a solvable Lie algebra L is the maximal nilpotent ideal of L. For a given solvable Lie algebra L its nilradical $NR(L)$ is unique and its dimension satisfies

$$\dim NR(L) \geqslant \tfrac{1}{2} \dim L. \tag{2.1}$$

A Lie algebra L is decomposable if it can, by change of basis, be transformed into a direct sum of two (or more) Lie algebras

$$L = L_1 \oplus L_2 \qquad [L_1, L_2] = 0. \tag{2.2}$$

It is indecomposable otherwise.

A set of matrices $\{X_i\}$ is linearly nilindependent if no non-zero linear combination of them is nilpotent, i.e.

$$X = \sum_{i=1}^{N} c_i X_i \qquad X^n = 0 \text{ implies } c_i = 0 \ \forall i. \tag{2.3}$$

An element n of a Lie algebra L is nilpotent in L, if it satisfies

$$[n \ldots [n[n, x]]] = 0 \qquad \forall x \in L. \tag{2.4}$$

A set of elements of L is linearly nilindependent if no non-zero linear combination of them is a nilpotent element in L.

2.2. *Basic classification theorem*

Let us consider the $2n + 1$-dimensional Heisenberg algebra $H(n)$ in its standard basis $\{P_1, \ldots, P_n, B_1, \ldots B_n, H\}$ with commutation relations

$$[P_i, B_k] = \delta_{ik} H \qquad [P_i, H] = [B_i, H] = 0. \tag{2.5}$$

We wish to extend this algebra to an indecomposable solvable Lie algebra $L(n, f)$ of dimension $2n + 1 + f$ having $H(n)$ as its nilradical. This means we wish to add f further linearly nilindependent elements to $H(n)$. Let us denote them $\{S_1, \ldots, S_f\}$.

The derived algebra of a solvable Lie algebra is contained in its nilradical [9]. The commutation relations of $L(n, f)$ involving the new elements S_α, will have the form

$$\begin{pmatrix} [S_\alpha, H] \\ [S_\alpha, P] \\ [S_\alpha, B] \end{pmatrix} = \begin{pmatrix} 2a_\alpha & \sigma^{\mathrm{T}}_{\alpha,1} & \sigma^{\mathrm{T}}_{\alpha,2} \\ \rho_{\alpha,1} & a_\alpha I_n + A_\alpha & C_\alpha \\ \rho_{\alpha,2} & D_\alpha & a_\alpha I_n + E_\alpha \end{pmatrix} \begin{pmatrix} H \\ P \\ B \end{pmatrix} \tag{2.6}$$

$$a_\alpha \in F, \sigma_{\alpha 1}, \sigma_{\alpha 2}, \rho_{\alpha 1} \qquad \rho_{\alpha 2}, \in F^{n\times 1} \qquad \alpha = 1, \ldots, f$$

$$A_\alpha, C_\alpha, D_\alpha, E_\alpha \in F^{n\times n}$$

$$P^{\mathrm{T}} = (P_1, \ldots P_n) \qquad B^{\mathrm{T}} = (B_1, \ldots B_n).$$

The superscript T denotes complex conjugation and the constants a_α were split off from A_α and E_α for future convenience. Further, we have

$$[S_\alpha, S_\beta] = r_{\alpha\beta} H + \gamma^i_{\alpha\beta} P_i + \mu^i_{\alpha\beta} B_i \qquad r_{\alpha\beta}, \gamma^i_{\alpha\beta}, \mu^i_{\alpha\beta} \in F. \tag{2.7}$$

We first change basis to put $\tilde{S}_\alpha = S_\alpha + \rho_{\alpha 1,i} B_i - \rho_{\alpha 2,i} P_i$. This amounts to setting $\rho_{\alpha 1} = \rho_{\alpha 2} = 0$ in (2.6).

Let us now impose the Jacobi identities. From the triplets $\{S_\alpha, P_i, H\}$ and $\{S_\alpha, B_i, H\}$ we obtain $\sigma_{\alpha 1} = \sigma_{\alpha 2} = 0$ in (2.6). From the triplets $\{S_\alpha, P_i, P_k\}$, $\{S_\alpha, B_i, B_k\}$, $\{S_\alpha, B_i, P_k\}$ we find that the remaining matrices in (2.6) satisfy

$$E_\alpha = -A^{\mathrm{T}}_\alpha \qquad C_\alpha = C^{\mathrm{T}}_\alpha \qquad D_\alpha = D^{\mathrm{T}}_\alpha. \tag{2.8}$$

It follows that the matrices

$$X_\alpha = \begin{pmatrix} A_\alpha & C_\alpha \\ D_\alpha & -A_\alpha^{\mathrm{T}} \end{pmatrix} \tag{2.9a}$$

belong to the symplectic Lie algebra $\mathrm{sp}(2n, F)$

$$X_\alpha K + K X_\alpha^{\mathrm{T}} = 0 \qquad K = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.9b}$$

Taking suitable linear combinations of the elements S_α we can arrange to have $a_1 = 1$, or 0, and $a_2 = \ldots = a_f = 0$.

The Jacobi identities for the triplets $\{S_\alpha, S_\beta, P_i\}$ and $\{S_\alpha, S_\beta, B_i\}$ imply $\mu^i_{\alpha\beta} = \gamma^i_{\alpha\beta} = 0$. Further they imply that the matrices X_α must commute. For $a_1 = 0$ they must also be linearly nilindependent, otherwise the nilradical would be larger than $H(n)$. For $a_1 = 0$ the matrices $\{X_2, \ldots X_f\}$ must be linearly nilindependent for the same reason, though X_1 may be nilpotent or even vanish. This imposes a restriction on the number of elements S_α that can be added. Indeed, the number of linearly nilindependent matrices $X_\alpha \in \mathrm{sp}(2n, F)$ is less or equal to the rank n of $\mathrm{sp}(2n, F)$ [17].

The remaining Jacobi identities for $\{S_1, S_\alpha, S_\beta\}$ (for $f \geqslant 3$) imply that for $a_1 = 1$ we have $r_{\alpha\beta} = 0$, $\alpha \neq 1$, $\beta \neq 1$. Redefining S_α in this case as $\tilde{S}_\alpha = S_\alpha - r_{1\alpha} H$ we obtain $r_{1\alpha} = 0$ as well.

The obtained commutation relations can be further simplified by transformations that respect the commutation relations in the nilradical $H(n)$ and the simplifications already achieved. We put

$$\xi = \begin{pmatrix} P \\ B \end{pmatrix} \qquad \xi' = G\xi. \tag{2.10}$$

The commutation relations (2.5) are written as

$$[\xi_a, \xi_b] = K_{ab} H \qquad 1 \leqslant a, b \leqslant 2n. \tag{2.11}$$

The transformation (2.10) must then satisfy

$$GKG^{\mathrm{T}} = K \tag{2.12}$$

with K as in (2.9), i.e. $G \in Sp(2n, F)$ belongs to the symplectic Lie group.

A further allowed transformation is the scaling

$$P_i' = \lambda P_i \qquad B_i' = \lambda B_i \qquad H' = \lambda^2 H \qquad \lambda \in F \qquad \lambda \neq 0. \tag{2.13}$$

We have thus proven the following result.

Theorem 1. Every indecomposable solvable Lie algebra $L(n,f)$ (over the field $F = \Re$ or $F = C$, containing the Heisenberg algebra $H(n)$ as its nilradical, can be written in a canonical basis $\{S_1, \ldots, S_f, P_1, \ldots, P_n, B_1, \ldots, B_n, H\}$ with commutation relations (2.5), supplemented by

$$\begin{pmatrix} [S_\alpha, H] \\ [S_\alpha, \xi] \end{pmatrix} = M_\alpha \begin{pmatrix} H \\ \xi \end{pmatrix} \qquad M_\alpha = \begin{pmatrix} 2a_\alpha & 0 \\ 0 & a_\alpha I_{2n} + X_\alpha \end{pmatrix} \tag{2.14}$$

$$[S_\alpha, S_\beta] = r_{\alpha\beta} H \qquad \alpha, \beta = 1, \ldots, f. \tag{2.15}$$

The vector column ξ is defined as

$$\xi^{\mathrm{T}} = \{P_1, \ldots P_N, B_1, \ldots, B_N\}.$$

The constants a_α satisfy

$$a_1 = \begin{cases} 1 \\ 0 \end{cases} \qquad a_2 = \ldots = a_f = 0. \tag{2.16}$$

The matrices $X_1, \ldots X_f$ satisfy (2.9) and

$$[X_\alpha, X_\beta] = 0. \tag{2.17}$$

For $a_1 = 0$, or $a_1 = 1$ the sets $\{X_1, \ldots X_f\}$, or $\{X_2, \ldots X_f\}$ are linearly nilindependent, respectively.

The constants $r_{\alpha\beta}$ satisfy

$$\begin{aligned} r_{\alpha\beta} &= -r_{\beta\alpha} = 0 && \text{for } a_1 = 1 \\ r_{\alpha\beta} &= -r_{\beta\alpha} \in F && \text{for } a_1 = 0. \end{aligned} \tag{2.18}$$

The dimension of the Lie algebra $L(n,f)$ is

$$\dim L(n,f) = 2n + 1 + f \qquad 0 \leqslant f \leqslant n+1. \tag{2.19}$$

The maximal value $f = n+1$ is achieved precisely if we have $a_1 = 1$, $X_1 = 0$ and $\{X_2, \ldots, X_f\}$ is a Cartan subalgebra of $Sp(2n, F)$. We then also have $r_{\alpha\beta} = 0$ for all α, β.

Two algebras $L(n,f)$ and $L'(n,f)$, characterized by $(a_1, X_\alpha, r_{\alpha\beta})$ and $(a_1', X_\alpha', r_{\alpha\beta}')$ are equivalent if, after an allowed transformation we have

$$X_\alpha = X_\alpha' \qquad a_1 = a_1' \qquad r_{\alpha\beta} = r_{\alpha\beta}'. \tag{2.20}$$

Allowed transformations are the symplectic transformations (2.12), the scaling (2.13) and linear combinations of S_α, respecting the form of (2.14).

A classification of the solvable algebras $S(f)$ thus boils down to a classification of Abelian subalgebras of $\mathrm{sp}(2n, F)$, containing no nilpotent elements.

Elements of $\mathrm{sp}(2n, \Re)$ and $\mathrm{sp}(2n, C)$ have been classified [18, 19], as have maximal Abelian subalgebras [17].

2.3. The lowest dimensional case: $n = 1$

The Heisenberg algebra $H(1)$ is three dimensional with basis $\{P, B, H\}$. Following theorem 1, we can extend $H(1)$ either by one element S, or by two commuting elements $\{S_1, S_2\}$. Let us consider these two cases separately.

(i) dim $L = 4$. ($f = 1$.) The algebra sp$(2, C)$ has two types of elements, nilpotent and non-nilpotent. The algebra sp$(2, R)$ has three types: nilpotent, compact and non-compact. The algebra L is completely characterized by one matrix M, figuring in the commutation relation (2.14).

For $F = \Re$ five inequivalent possibilities occur, characterized by the matrices

$$M^{(1)} = \begin{pmatrix} 2 & & \\ & 1+b & 0 \\ & 0 & 1-b \end{pmatrix} \qquad b \geqslant 0 \qquad M^{(2)} = \begin{pmatrix} 2 & & \\ & 1 & 1 \\ & 0 & 1 \end{pmatrix}$$
$$M^{(3)} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \tag{2.21a}$$

$$M^{(4)} = \begin{pmatrix} 2 & & \\ & 1 & a \\ & -a & 1 \end{pmatrix} \qquad a > 0 \qquad M^{(5)} = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}. \tag{2.21b}$$

For $F = C$ precisely three possibilities occur, namely those of (2.21*a*). Indeed, over C, $M^{(4)}$ is equivalent to $M^{(1)}$, $M^{(5)}$ to $M^{(3)}$.

(ii) dim $L = 5$. ($f = 2$.) In this case we must have

$$[S_1, S_2] = 0$$

and the action of S_1 and S_2 on $H(2)$ is characterized by a matrix pair $\{M_1, M_2\}$. Over $\Re$ two possibilities occur

$$M_1^{(1)} = \begin{pmatrix} 2 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad M_2^{(1)} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \tag{2.22a}$$

and

$$M_1^{(2)} = \begin{pmatrix} 2 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad M_2^{(2)} = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}. \tag{2.22b}$$

over $F = C$ the two cases are equivalent and only one survives, the natural choice being (2.22*a*).

2.4. The case: $n = 2$

The Heisenberg algebra $H(2)$ is five-dimensional; we have $0 \leqslant k \leqslant 3$, i.e. we can add one, two or three elements S_α.

(iii) dim $L = 6$. Algebras $L(2, 1)$. For $F = C$, 8 types of such algebras exist, one of them depending on 2 complex parameters, 3 depending on one complex parameter, 4 without parameters.

For $F = \Re$ altogether 19 types exist, among them 4 depend on 2 real parameters, 8 on one real parameter, 7 without parameters.

They are listed in table A1 of the appendix.

(iv) dim $L = 7$. Algebras $L(2,2)$. For $F = C$, 8 types of such algebras exist, one depending on 2 complex parameters, 2 on one parameter, 5 without parameters.

For $F = \Re$, 27 such types exist, 4 depending on two parameters, 9 on one and 14 without parameters. They are presented in table A2 of the appendix.

(v) dim $L = 8$. Algebras $L(2,3)$. The commutation relations for S_1, S_2 and S_3 are

$$[S_\alpha, S_\beta] = 0 \qquad \alpha, \beta = 1,2,3. \tag{2.23}$$

The action of S_α on $\{H, P_1, P_2, B_1, B_2\}$ is given by three matrices M_α as in (2.14). For $F = C$ the Cartan subalgebra of sp$(4, C)$ is unique and we have

$$M_1^{(1)} = \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \qquad M_2^{(1)} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & 0 \end{pmatrix}$$

$$M_3^{(1)} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & -1 \end{pmatrix} \tag{2.24}$$

For sp$(4, \Re)$ we have four inequivalent Cartan subalgebras [17], so in addition to the algebra L characterized by the matrices (2.24) we have three more cases, characterized by

$$M_1^{(2)} = \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \qquad M_2^{(2)} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}$$

$$M_3^{(2)} = \begin{pmatrix} 0 & & & & \\ & 0 & 1 & & \\ & -1 & 0 & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \tag{2.25}$$

$$M_1^{(3)} = \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \qquad M_2^{(3)} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & 0 \end{pmatrix}$$

$$M_3^{(3)} = \begin{pmatrix} 0 & & & & \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \\ & 0 & -1 & 0 & 0 \end{pmatrix} \tag{2.26}$$

$$M_1^{(4)} = \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \qquad M_2^{(4)} = \begin{pmatrix} 0 & & & & \\ & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \\ & -1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_3^{(4)} = \begin{pmatrix} 0 & & & & \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \\ & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{2.27}$$

3. Invariants of the solvable Lie algebras

3.1. Comments on Casimir invariants

In theorem 1 we have characterized the indecomposable solvable Lie algebras $L(nf)$ with the Heisenberg algebras $H(n)$ as nilradicals. The corresponding Lie group $G(n,f)$ acts on $L(n,f)$ via its adjoint representation. The group $G(n,f)$ acts on the dual L^* of $L(n,f)$ via its coadjoint representation and sweeps out orbits in L^*. These orbits are characterized by certain invariants I_μ. If they are polynomials, they give rise to Casimir operators: elements of the centre of the enveloping algebra of $L(n,f)$. More generally the invariants in the coadjoint representation may correspond to functions of the infinitesimal operators of $L(n,f)$. In particular these may be rational functions, or even algebraic, or transcendental functions [13,20]. We shall call the corresponding operators 'Generalized Casimir Invariants'.

In any case the invariants of the coadjoint representation characterize irreducible representations of the considered Lie algebra [16,21] and are hence of considerable importance in physical applications.

We shall now calculate a basis of the invariants of the group $G(n,f)$, using Lie algebraic methods [13,20].

We shall realize the coadjoint representation of $L(n,f)$ in a space of differentiable functions of $2n+1+f$ variables which we denote $\{s_1,\ldots,s_f,p_1,\ldots p_n,b_1,\ldots,b_n,h\}$. The algebra $L(n,f)$ of theorem 1 will be realized by differential operators, the form of which can be read off from the commutation relations (2.5), (2.14) and (2.15). Namely, we have

$$H = -2a_1 h\partial_{s_1} \tag{3.1a}$$

$$P_i = h\partial_{b_i} - (A_{ia}^{\alpha} p_a + C_{ia}^{\alpha} b_a)\partial_{s_\alpha} - a_1 p_i \partial_{s_1}$$

$$B_i = -h\partial_{p_i} - (D_{ia}^{\alpha} p_a - A_{ai}^{\alpha} b_a)\partial_{s_\alpha} - a_1 b_i \partial_{s_1} \tag{3.1b}$$

$$S_\alpha = 2a_\alpha h\partial_h + (A_{ia}^{\alpha} p_a + C_{ia}^{\alpha} b_a + a_\alpha p_i)\partial_{p_i} + (D_{ia}^{\alpha} p_a - A_{ai}^{\alpha} b_a + a_\alpha b_i)\partial_{b_i} + r_{\alpha\beta} h\partial_{s_\beta} \tag{3.1c}$$

$$i = 1,\ldots,n \qquad \alpha = 1,\ldots,f \qquad a_1 = \begin{cases} 1 \\ 0 \end{cases} \qquad a_\alpha = 0 \qquad \alpha \geqslant 2$$

$r_{\alpha\beta=0}$ for $a_1 = 1$.

The matrices A, $C = C^{\mathrm{T}}$ and $D = D^{\mathrm{T}}$ are the same as in the commutation relations (2.14), i.e. in (2.8).

The invariants of $G(n,f)$ will be obtained as functionally independent solutions of the system of first-order linear partial differential equations

$$HF = 0 \qquad P_i F = 0 \qquad B_i F = 0 \qquad S_\alpha F = 0 \qquad F = F(s_\alpha, p_i, b_i, h). \tag{3.1d}$$

3.2. The Casimir invariants of the $L(n,f)$ algebras

Let us treat the cases $a_1 = 1$ and $a_1 = 0$ separately.

Theorem 2. The indecomposable solvable Lie algebra $L(n,f)$ with commutation relations (2.5), (2.14), (2.15) and $a_1 = 1$ (and hence $r_{\alpha\beta} = 0$) has precisely $(f-1)$ generalized Casimir invariants. They can be written in the form

$$I_{\alpha-1} = [2HS_\alpha + A^\alpha_{ij}(B_i P_j + P_j B_i) + C^\alpha_{ij} B_i B_j - D^\alpha_{ij} P_i P_j] H^{-1}$$
$$\alpha = 2, \ldots, f. \tag{3.2}$$

Proof. Applying H for $a_1 = 1$ to the function $F(s_\alpha, p_i, b_i, h)$ we find that F is independent of s_1. Applying the basis operators of $H(n)$, namely P_i and B_i of (3.1) successively to F and using the method of characteristics, we find that F can depend only on $Z_1 = h$ and Z_α, $2 \leqslant \alpha \leqslant f$ with

$$Z_\alpha = 2hs_\alpha + 2A^\alpha_{ij} b_i p_j + C^\alpha_{ij} b_i b_j - D^\alpha_{ij} p_i p_j . \tag{3.3}$$

Let us now apply the remaining operators S_β to $F(h, Z_2, \ldots Z_f)$. We have

$$S_\beta Z_\alpha = (A^\alpha A^\beta - A^\beta A^\alpha + C^\alpha D^\beta - C^\beta D^\alpha)_{ik} b_i p_k + (D^\beta A^\alpha - D^\alpha A^\beta)_{ik} p_i p_k$$
$$+ (A^\alpha C^\beta - A^\beta C^\alpha)_{ik} b_i b_k \qquad 2 \leqslant \alpha, \beta \leqslant f. \tag{3.4}$$

The commutativity conditions (2.17) imply that the first term in (3.4) vanishes. The second and third term also vanish, since the symmetric parts of the matrices $D^\beta A^\alpha - D^\alpha A^\beta$ and $A^\alpha C^\beta - A^\beta C^\alpha$ are zero as a result of the same commutativity relations. The antisymmetric parts, when contracted with the symmetric tensor $p_i p_k$ (or $b_i b_k$, respectively), also vanish. Hence we have $S_\beta Z_\alpha = 0$ for $\beta \geqslant 2$, $\alpha \geqslant 2$. Finally, we apply S_1 and obtain

$$S_1 F(h, Z_\alpha) = 2(hF_h + Z_2 F_{Z_2} + \ldots + Z_k F_{Z_k}) = 0. \tag{3.5}$$

From (3.5) we conclude that F is an arbitrary function of the invariants (3.2) and this proves theorem 2.

Theorem 3. The indecomposable solvable Lie algebra $L(n, f)$ with commutation relations (2.5), (2.14), (2.15) and $a_1 = 0$ has N Casimir invariants

$$I_1 = H \qquad I_r = \sum_{\alpha=1}^{f} a_{\mu\alpha} Z_\alpha \qquad 2 \leqslant \mu \leqslant N \tag{3.6}$$

where

$$\begin{aligned} &N = f + 1 - r \qquad r = \text{rank } R \\ &R = \{r_{\alpha\beta}\} \qquad r_{\alpha\beta} = -r_{\beta\alpha} \qquad 1 \leqslant \alpha, \beta \leqslant f \end{aligned} \tag{3.7}$$

$$\begin{aligned} Z_\alpha = 2H S_\alpha &+ \sum_{i,j=1}^{n} A_{ij}^{\alpha}(B_i P_j + P_j B_i) + \sum_{i,j=1}^{n} C_{ij}^{\alpha} B_i B_j \\ &- \sum_{i,j=1}^{n} D_{ij}^{\alpha} P_i P_j \qquad 1 \leqslant \alpha \leqslant f. \end{aligned} \tag{3.8}$$

The constants $a_{\mu\alpha}$ are determined from the condition that the column vectors a_μ span the nullspace of the matrix R:

$$R a_\mu = 0 \qquad 2 \leqslant \mu \leqslant N. \tag{3.9}$$

Proof. From the form of the operators (3.1) with $a_1 = 0$ we see directly that h is an invariant. The equations $P_i B = B_i F = 0$ imply $F = F(h, Z_1, \ldots Z_f)$ with Z_α as in (3.3), and $1 \leqslant \alpha \leqslant f$. Following the same reasoning as in the proof of theorem 2 we obtain

$$S_\alpha F = 2h^2 r_{\alpha\beta} \frac{\partial F}{\partial Z_\beta} = 0. \tag{3.10}$$

It follows that the elementary solutions of (3.10) are linear combinations of Z_β with constant coefficients, as in (3.6). We then have

$$S_\alpha I_\mu = h^2 r_{\alpha\beta} a_{\mu\beta} = 0$$

i.e. we obtain (3.9) and this completes the proof of theorem 3.

3.3. Examples

To illustrate theorem 2, let us consider the eight-dimensional Lie algebra $L(2, 3)$, characterized by the matrices of (2.24). Applying the theorem, or calculating the invariants directly, we obtain

$$I_1 = \frac{2H S_2 + P_1 B_1 + B_1 P_1}{H} \qquad I_2 = \frac{2H S_3 + P_2 B_2 + B_2 P_2}{H}. \tag{3.11}$$

To illustrate theorem 3, let us consider the case of the algebras $L(n,3)$ with n arbitrary. The matrix $\Re$ in this case is

$$R = \begin{pmatrix} 0 & r_{12} & r_{13} \\ -r_{12} & 0 & r_{23} \\ -r_{13} & -r_{23} & 0 \end{pmatrix} \tag{3.12}$$

and its rank is either $r = 0$ or $r = 2$.

For $r = 0$ we have $r_{12} = r_{23} = r_{31} = 0$ and we obtain 4 invariants

$$\{h, Z_1, Z_2, Z_3\} \tag{3.13a}$$

with Z_i as in (3.8).

In all other cases we have $r = 2$ and the two invariants are

$$I_1 = H \qquad I_2 = r_{23}Z_1 + r_{31}Z_2 + r_{12}Z_3. \tag{3.13b}$$

As a further illustration of theorem 3, consider the Lie algebras $L(n,4)$. The matrix $R \in F^{4\times 4}$ can have rank 0, 2 or 4. The invariants are respectively

$$r = 0 \qquad \{H, Z_1, Z_2, Z_3, Z_4\} \tag{3.14a}$$

$$r = 2 \qquad \{H, r_{23}Z_1 + r_{31}Z_2 + r_{12}Z_3, r_{24}Z_1 + r_{41}Z_2 + r_{12}Z_4\} \tag{3.14b}$$

$$r = 4 \qquad \{H\}. \tag{3.14c}$$

4. Conclusions

Theorem 1 reduces the problem of finding all indecomposable solvable Lie algebras $L(n,f)$ with $H(n)$ as their nilradical, to the problem of classifying all Abelian subalgebras of $\mathrm{sp}(2n,F)$, containing no nilpotent elements. The classification group is $Sp(2N,F)$, figuring in physical applications as the group of canonical transformations leaving the Heisenberg relation invariant [22–25].

Theorems 2 and 3 provide all the invariants of the coadjoint action of the corresponding groups $G(n,f)$, or in more physical terms, the Casimir operators and the generalized Casimir operators.

The results of this article should be of use e.g. in constructing the unitary representations of the solvable Lie algebras $L(n,f)$ and Lie groups $G(n,f)$, in particular in the calculation of the characters of these representations.

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Table A1. The Lie algebras $L(2,1)$.

Name	X	$a=1$	$a=0$
$F_1(1,b,c)$ $F_2(0,1,c)$	$\begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -c \end{pmatrix}$	$\mathcal{C}$: $\lvert c\rvert \leqslant \lvert b\rvert$ $0 \leqslant \arg(b), \arg(c) \leqslant \pi$ For $\lvert b\rvert = \lvert c\rvert$ $\arg(c) < \arg(b)$ $\Re$: $0 \leqslant c \leqslant b$	$\mathcal{C}$: $b=1, 0 \leqslant \lvert c\rvert \leqslant 1$ $\Re$: $b=1, 0 \leqslant c \leqslant 1$
$F_3(1)$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$		N.A.
$F_4(1,b)$ $F_5(0,1)$	$\begin{pmatrix} b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\mathcal{C}$: $0 \leqslant \arg(b) \leqslant \pi$ $\Re$: $0 \leqslant b$	$\mathcal{C}$: $b=1$ $\Re$: $b=1$
$F_6(1,b)$ $F_7(0,1)$	$\begin{pmatrix} b & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & -1 & -b \end{pmatrix}$	$\mathcal{C}$: $0 \leqslant \arg(b) \leqslant \pi$ $\Re$: $0 \leqslant b$	$\mathcal{C}$: $b=1$ $\Re$: $b=1$
$F_8(1)$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$		N.A.
$R_9(1,b,c)$ $R_{10}(0,b,1)$	$\begin{pmatrix} b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & -b & 0 \\ 0 & -c & 0 & 0 \end{pmatrix}$	$0 < b,\ c \neq 0$ or $b=0,\ c>0$	$c=1, 0 \leqslant b$
$R_{11}(1,b)$ $R_{12}(0,1)$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{pmatrix}$	$b \neq 0$	$b=1$
$R_{13}(1,b,c)$ $R_{14}(0,1,c)$	$\begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ -b & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \end{pmatrix}$	$-b \leqslant c \leqslant b$ $c \neq 0$	$b=1$ $0 < \lvert c\rvert \leqslant 1$
$R_{15}(1,b,c)$ $R_{16}(0,b,1)$	$\begin{pmatrix} b & c & 0 & 0 \\ -c & b & 0 & 0 \\ 0 & 0 & -b & c \\ 0 & 0 & -c & -b \end{pmatrix}$	$0 < c$ $0 \leqslant b$	$c=1$ $b \geqslant 0$
$R_{17}(1,b)$ $R_{18}(0,1)$	$\begin{pmatrix} 0 & b & 1 & 0 \\ -b & 0 & 0 & 1 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}$	$0 < b$	$b=1$
$R_{19}(1)$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$		N.A.

Appendix. Extensions of $H(2)$ by one and two elements

As an illustration of theorem 1, we shall give a list of all algebras $L(2,1)$ and $L(2,2)$. The algebras $L(2,3)$ were presented in section 2.

Let us first consider the case $L(2,1)$. We characterize the algebra by the constant $a = 1$ or 0 and by the matrix $X \in \mathrm{sp}(4, F)$ figuring in theorem 1. For each algebra we introduce a name $F_i(a,b,c)$ or $R_i(a,b,c)$. The letter F indicates that the algebra exists both for $F = C$ and $F = \Re$; the algebras R exist over $\Re$ only (i.e. over C they are equivalent to the algebras F). The subscript simply enumerates the algebras. The label a in the brackets takes values $a = 1$ or $a = 0$. The other labels indicate parameters in the matrix X. If there are less than two parameters we drop corresponding labels. In table A1, we give a list of all inequivalent algebras $L(2,1)$ over $\Re$ and C. In the last two columns we give the range of parameters b, c separately for $a = 1$ and $a = 0$ and for $F = C$ and $\Re$. The letters N.A. in column 4 indicate that the algebra does not exist for $a = 0$.

The statement is: any Lie algebra $L(2,1)$ is equivalent to precisely one listed in table A1.

We now turn to the algebras $L(2,2)$ characterized by two constants, a_1 and $r = r_{12}$, and two commuting matrices $X_1, X_2 \in \mathrm{sp}(4, F)$.

Any such algebra is equivalent to precisely one in table A2. The first 8 algebras correspond to $a_1 = a_2 = 0$ and $r = 0$ or 1, as indicated in columns 1 and 2. The remaining ones correspond to $a_1 = 1$, $a_2 = 0$, $r = 0$.

The notation $F_i(b,c)$ indicates that the algebras exist and are mutually inequivalent over both $F = C$ and $F = \Re$; the notation $R_i(b,c)$ indicates algebras that differ from those denoted F_i only over the field of real numbers $\Re$. The range of parameters over C and $\Re$ is given in the last column of table A2.

Table A2. The Lie algebras $L(2,2)$.

a_1	r	Name	X_1	X_2	Range of parameters
0	1 0	F_1 F_2	$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix}$	
0	1 0	R_9 R_{10}	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$	$\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$	
0	1 0	R_{11} R_{12}	$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}$	
0	1 0	R_{13} R_{14}	$\begin{pmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}$	

Table A2. (continued)

a_1	r	Name	X_1	X_2	Range of parameters
1	0	$F_3(b,c)$	$\begin{pmatrix}0&0&0&0\\0&b&0&0\\0&0&0&0\\0&0&0&-b\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&c&0&0\\0&0&-1&0\\0&0&0&-c\end{pmatrix}$	$\mathcal{C}$: $0\leqslant \arg(b)<\pi$, $\lvert c\rvert\leqslant 1$ $\Re$: $b\geqslant 0$, $-1\leqslant c\leqslant 1$
1	0	F_4	$\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$	
1	0	F_5	$\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$	
1	0	F_6	$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}$	
1	0	$F_7(b)$	$\begin{pmatrix}-b&0&0&0\\0&0&0&0\\0&0&b&0\\0&0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&-1&0\\0&0&0&0\end{pmatrix}$	$\mathcal{C}$: $0\leqslant \arg(b)<\pi$ $\Re$: $0\leqslant b$
1	0	$F_8(b)$	$\begin{pmatrix}b&0&0&0\\0&b&0&0\\0&0&-b&0\\0&0&0&-b\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&-1&0\\0&0&-1&-1\end{pmatrix}$	$\mathcal{C}$: $0\leqslant \arg(b)<\pi$ $\Re$: $0\leqslant b$
1	0	$R_{15}(b)$	$\begin{pmatrix}0&b&0&0\\-b&0&0&0\\0&0&0&b\\0&0&-b&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$	$0<b$
1	0	$R_{16}(b)$	$\begin{pmatrix}0&0&0&b\\0&0&b&0\\0&-b&0&0\\-b&0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$	$0<b$
1	0	$R_{17}(b)$	$\begin{pmatrix}0&0&0&0\\0&0&0&b\\0&0&0&0\\0&-b&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}$	$b\neq 0$
1	0	$R_{15}(b)$	$\begin{pmatrix}0&b&0&0\\-b&0&0&0\\0&0&0&b\\0&0&-b&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$	$0<b$
1	0	$R_{16}(b)$	$\begin{pmatrix}0&0&0&b\\0&0&b&0\\0&-b&0&0\\-b&0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$	$0<b$

Table A2. (continued)

a_1	r	Name	X_1	X_2	Range of parameters
1	0	$R_{17}(b)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$b \neq 0$
1	0	$R_{15}(b)$	$\begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$0 < b$
1	0	$R_{16}(b)$	$\begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \\ 0 & -b & 0 & 0 \\ -b & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$0 < b$
1	0	$R_{17}(b)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$b \neq 0$
1	0	$R_{18}(b, c)$	$\begin{pmatrix} b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -c & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$0 \leqslant b$
1	0	$R_{19}(b)$	$\begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$b \neq 0$
1	0	R_{20}	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	
1	0	$R_{21}(b, c)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c \\ -1 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \end{pmatrix}$	$-1 \leqslant c \leqslant 1$ $c \neq 0,\ b \in \Re$
1	0	$R_{22}(b)$	$\begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$0 < b$
1	0	$R_{23}(b)$	$\begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$b \in \Re$
1	0	$R_{24}(b, c)$	$\begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}$	$\begin{pmatrix} c & 1 & 0 & 0 \\ -1 & c & 0 & 0 \\ 0 & 0 & -c & 1 \\ 0 & 0 & -1 & -c \end{pmatrix}$	$0 \leqslant b$ $0 \leqslant c$

Table A2. (continued)

a_1	r	Name	X_1	X_2	Range of parameters
1	0	$R_{25}(b)$	$\begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ -b & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$b \neq 0$
1	0	R_{26}	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	
1	0	$R_{27}(b)$	$\begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$b \in \Re$

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